Exercise 48

Use logarithmic differentiation to find the derivative of the function.

$$y = (\sin x)^{\ln x}$$

Solution

Take the natural logarithm of both sides and use the properties of logarithms to simplify the right side.

$$\begin{aligned}\ln y &= \ln(\sin x)^{\ln x} \\ &= (\ln x)\ln\sin x\end{aligned}$$

Differentiate both sides with respect to x.

$$\begin{aligned}\frac{d}{dx}(\ln y) &= \frac{d}{dx}[(\ln x)\ln\sin x] \\ \frac{1}{y}\cdot\frac{d}{dx}(y) &= \left[\frac{d}{dx}(\ln x)\right]\ln\sin x + (\ln x)\left[\frac{d}{dx}(\ln\sin x)\right] \\ \frac{1}{y}\cdot\frac{dy}{dx} &= \left(\frac{1}{x}\right)\ln\sin x + (\ln x)\left[\frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x)\right] \\ \frac{1}{y}\frac{dy}{dx} &= \frac{\ln\sin x}{x} + (\ln x)\left[\frac{1}{\sin x}\cdot(\cos x)\right] \\ \frac{dy}{dx} &= y\left(\frac{\ln\sin x}{x} + \ln x\cot x\right) \\ &= (\sin x)^{\ln x}\left(\frac{\ln\sin x}{x} + \ln x\cot x\right)\end{aligned}$$